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THE INDENTATION OF A FLAT PUNCH INTO AN IDEAL RIGID-PLASTIC HALF-SPACE UNDER THE ACTION OF SHEAR CONTACT STRESSES†

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A general plane problem of the impression of a flat punch into a rigid-plastic half-space under the action of transverse and longitudinal shear contact stresses is considered. The condition of complete plasticity and the hyperbolic equations of the general plane problem of the theory of ideal plasticity [1] are used. The reduction of the limit pressure on the punch is determined as a function of the shear contact stresses. © 2002 Elsevier Science Ltd. All rights reserved.

1. BASIC EQUATIONS

We solve the problem of the indentation of a flat punch into a right-plastic half-space under the action of longitudinal and transverse shear contact stresses on the punch boundary using the condition of complete plasticity which, in the principal stresses, has the form

$$\sigma_1 = \sigma_2, \quad \sigma_3 = \sigma_1 + 2k \tag{1.1}$$

where k is the shear yield point of the right-plastic material. The condition of complete plasticity ensures considerable freedom of the plastic flow, leads to quasilinear hyperbolic differential equations with an efficient algorithm for solving boundary-value problems, including discontinuities and singular points, and corresponds to the shear mechanism of the plastic flow of solids [2].

Below, we shall employ dimensionless stresses, taking $2k = 1$ as the unit of stress and the width of the punch as the unit of length. We write the mean stress σ in the case of condition (1.1) in the form

$$\sigma = \sigma_1 + 1/3 \tag{1.2}$$

We specify the direction cosines of the stress σ_3 with coordinate axes x, y, z by means of the angles θ and φ

$$n_1 = \cos(\theta/2)\cos\varphi, \; n_2 = \cos(\theta/2)\sin\varphi, \; n_3 = \sin(\theta/2) \tag{1.3}$$

The components of the stress tensor in Cartesian coordinates (x, y, z), which satisfy condition (1.1), can be expressed in terms of σ, θ and φ [1]

$$\begin{aligned} &\sigma_x = \sigma - 1/3 + 1/2(1+\cos\theta)\cos^2\varphi, \; \sigma_y = \sigma - 1/3 + 1/2(1+\cos\theta)\sin^2\varphi \\ &\sigma_z = \sigma - 1/3 + 1/2(1-\cos\theta), \; \tau_{xy} = 1/2(1+\cos\theta)\sin\varphi\cos\varphi \\ &\tau_{xz} = 1/2\sin\theta\cos\varphi, \; \tau_{yz} = 1/2\sin\theta\sin\varphi \end{aligned} \tag{1.4}$$

If the length of the punch in the direction of the z axis is significantly greater than its width in the direction of the x axis, it can be assumed that σ, θ and φ are independent of the z coordinate. This is the case of the general plane problem of the theory of ideal plasticity for which quasilinear equations of the hyperbolic type for the functions σ, θ and φ have been obtained, with three equations for the characteristics and differential relations along them

$$(dy/dx)_{\beta,\alpha} = \operatorname{tg}[\varphi \pm (\pi/4 + \mu)], \; \operatorname{tg} 2\mu = \xi_-(\theta) \tag{1.5}$$

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$$d\sigma \pm \xi_+(\theta)\, d\varphi = 0 \text{ along } \beta \text{ and } \alpha \tag{1.6}$$

$$(dy/dx)_\gamma = \operatorname{tg}\varphi \tag{1.7}$$

$$\zeta(\theta)d\sigma + \sin\theta \sin 2\varphi\, d\varphi + d\theta = 0 \text{ along } \gamma \tag{1.8}$$

Here

$$\xi_\pm(\theta) = (1 \pm \cos\theta)/(2\sqrt{\cos\theta}), \quad \zeta(\theta) = 2\sin\theta/(1+\cos\theta)$$

When $\theta = 0$, we obtain from expressions (1.4) $\tau_{xz} = \tau_{yz} = \pm\, 0$ and Eqs (1.5) define the orthogonal characteristics of plane strain with Hencky's relations (1.6). When $0 < \theta < \pi/2$, the characteristics (1.5) are non-orthogonal and the γ-characteristic (1.7) is the bisectrix of the angle between the α- and β-characteristics (1.5).

2. BOUNDARY CONDITIONS

We will assume that the normal stresses on the stress-free boundary of the half-space AC (Fig. 1) satisfy the condition of complete plasticity (1.1), that is, $\sigma_x = \sigma_z = \sigma_1 = \sigma_2 = -1$, $\sigma_y = \sigma_3 = 0$ and, from relations (1.2) and (1.3), we find

$$\sigma = -\tfrac{2}{3}, \quad \varphi = \pi/2, \quad \theta = 0 \quad \text{in } AC \tag{2.1}$$

When $\theta = 0$, the characteristics (1.5) are linear $(dy/dx)_{\alpha,\beta} = \pm 1$ and, in the region ABC, there is a uniform stressed state (2.1) for plane strain.

The characteristics (1.5) are only defined for positive values of $\cos\theta$ and therefore, on the contact boundary of the punch OA, we specify the angle θ as a parameter of the problem in the range

$$0 \leqslant \theta < \pi/2 \quad \text{in } OA \tag{2.2}$$

We specify the second parameter, the angle φ, which defines the shear contact stresses according to Eqs (1.4), in a range which depends on the angle θ

$$c\mu \leqslant \varphi < \pi/4 + \mu \quad \text{in } OA \tag{2.3}$$

$$0 \leqslant c \leqslant 1, \quad \mu = \tfrac{1}{2}\operatorname{arctg}\xi_-(\theta) \tag{2.4}$$

The upper limit in inequality (2.3) is determined by the degeneration of the field of the characteristics into a line tangential to the punch boundary, and the lower limit monitors the degeneration of the field of the characteristics at large angles θ, when the angle between the directions of the α- and β-characteristics approaches π.

At the singular point A, we find the change in the mean stress from the stress-free boundary AC to the punch boundary OA by integrating Eq. (1.6) over the degenerate α-characteristic

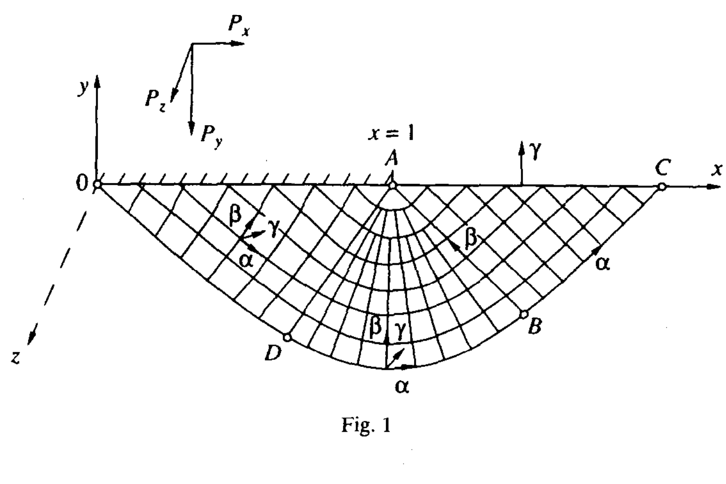

Fig. 1

$$\sigma = -\frac{2}{3} + \int\limits_{\pi/2}^{\varphi} \xi_+(\theta) d\varphi \tag{2.5}$$

We use a similar integral over the degenerate β-characteristic to monitor the support capability of a rigid wedge with vertex at the point O.

3. PLANE STRAIN

In the case of plane strain $\theta \equiv 0$. The orthogonal characteristics (1.5) in the region ABD form a centred fan with straight β-lines, and integral (2.5) determines the values of σ in the region of the uniform stressed state OAD

$$\sigma = -2/3 - (\pi/2 - \varphi) \tag{3.1}$$

We find the pressure on the punch and the shear contact stresses from expressions (1.4)

$$P_y = -\sigma_y = 1 + \pi/2 - (\varphi + \sin^2\varphi), \quad P_z = \tau_{yz} = 0, \quad P_x = \tau_{xy} = 1/2 \sin 2\varphi \tag{3.2}$$

For the length L of the characteristic AB and the coordinate x_c of the point C, which define the dimensions and shape of the field of the characteristics, we have

$$L = \sin(\pi/4 - \varphi), \quad x_c = 1 + \sqrt{2}\sin(\pi/4 - \varphi) \tag{3.3}$$

where φ varies over the range (2.3) when $\mu = 0$.

When $\varphi = 0$, we obtain a smooth Prandtl's punch, and the limit pressure has the form

$$P_y = 1 + \pi/2, P_x = P_z = 0; x_c = 2 \tag{3.4}$$

In this case, a wedge with vertex at the point O is loaded up to the plastic state and the field of the characteristics is symmetrical about the middle of the punch.

When $\varphi = \pi/4$, we obtain from relations (3.2) and (3.3)

$$P_y = 1/2(1 + \pi/2), \quad P_x = 1/2, \quad P_z = 0, \quad L = 0, \quad x_c = 1 \tag{3.5}$$

The field of the characteristics degenerates into a shear line, which coincides with the punch boundary, the shear contact stress in the direction of the x axis is equal to the shear yield point and the pressure on the punch is half the pressure on Prandtl's punch.

4. GENERAL PLANE STRAIN

When $\theta > 0$, we obtain the field of the characteristics and the stress field by numerical integration of Eqs (1.5)–(1.8) with boundary conditions (2.1)–(2.5).

At the regular mesh points of the characteristics, which do not belong to the singular point A and the boundary of the punch OA, we solve the Cauchy problem for the functions σ, φ and θ which are known at points *1* and *2* of the Cauchy contour (Fig. 2) by approximating the differentials in Eqs (1.5)–(1.8) by finite differences and the functions φ and θ by their mean values along the characteristics. The coordinates x, y of point P must satisfy the three differential equations of the characteristics which, in the finite-difference approximation, have the form

$$(y - y_1)/(x - x_1) = \operatorname{tg}[\varphi - (\pi/4 + \mu)] \text{ on } \alpha \tag{4.1}$$

$$(y - y_2)/(x - x_2) = \operatorname{tg}[\varphi + (\pi/4 + \mu)] \text{ on } \beta \tag{4.2}$$

$$(y - y_3)/(x - x_3) = \operatorname{tg}\varphi \text{ on } \gamma \tag{4.3}$$

We find the unknown coordinates of point *3* at the intersection of the γ-characteristic and the Cauchy contour, which we approximate by a chord between points *1* and *2*

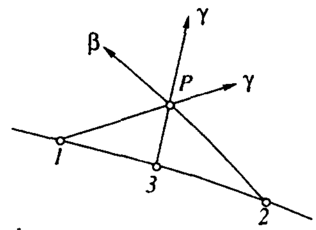

Fig. 2

$$(y_3-y_1)/(x_3-x_1) = (y_2-y_1)/(x_2-x_1) \tag{4.4}$$

For known coordinates of point *3*, we find the values of the functions σ, φ, θ at this point by linear interpolation between points *1* and *2*, at which these functions are defined.

The differential relations along the three characteristics take the form

$$\sigma-\sigma_1 = \xi_+(\theta)\,(\varphi-\varphi_1) \text{ on } \alpha \tag{4.5}$$

$$\sigma-\sigma_2 = -\xi_+(\theta)\,(\varphi-\varphi_2) \text{ on } \beta \tag{4.6}$$

$$\theta_3-\theta = \zeta(\theta)\,(\sigma-\sigma_3)+\sin\theta\,\sin 2\varphi\,(\varphi-\varphi_3) \text{ on } \gamma \tag{4.7}$$

where σ_1, σ_2, σ_3 are the values of σ at the points *1*, *2* and *3*.

Equations (4.1)–(4.7) contain the unknown coordinates x, y of the point P, the unknown functions σ, φ and θ at the point P and the unknown coordinates x_3, y_3. An iterative procedure is used to solve this system of equations.

We specify the initial values of the angles $\varphi=\varphi_1$, $\theta=\theta_1$ for the α-characteristic $1-P$, $\varphi=\varphi_2$, $\theta=\theta_2$ for the β-characteristic $2-P$ and $\varphi = \frac{1}{2}(\varphi_1+\varphi_2)$, $\theta=\frac{1}{2}(\theta_1+\theta_2)$ for the γ-characteristic $3-P$. The calculations are carried out using the following scheme.

1. We calculate x and y using Eqs (4.1) and (4.2), and σ and φ at the point P using Eqs (4.5) and (4.6).
2. Using Eqs (4.3) and (4.4), we calculate the coordinates x_3, y_3 of point 3 and, by linear interpolation between points *1* and *2*, we find the values of σ_3, φ_3, θ_3 at point 3.
3. Using Eq. (4.7), we calculate θ at the point P.
4. We calculate the mean values of φ and θ between the points $1-P$, $2-P$, $3-P$ and return to step 1.

An absolute difference in the successive values of φ and θ at the point P of the order of 10^{-5} is attained after two to three iterations.

We find the field of the characteristics ABD (Fig. 1) from the solution of the Goursat problem for known values of the functions σ, φ, θ in the β-characteristic AB and at the singular point A, by calculating the regular mesh points of the characteristics using Eqs (4.1)–(4.7). Then, in the region OAD, we solve the mixed problem with the known values of the functions σ, φ, θ for the β-characteristic AD and boundary conditions on AO. We find the x coordinates and the values of σ at the mesh points on the boundary OA from the linear equations (4.1) and (4.5), since φ and θ are known for OA.

The field of the characteristics in the region $OABD$ is determined, apart from the unknown length L of the characteristic AB, which we find from the condition that the coordinate x_0 is equal to zero at the point O. The algorithm for constructing the field of the characteristics defines x_0 as a continuous function of the parameter L which must satisfy the condition

$$x_0(L) = 0 \tag{4.8}$$

We solve Eq. (4.8) using Newton's iterative method, approximating the derivative with a finite-difference ratio and taking the length L for the plane strain (3.3) as the initial approximation

$$L_{i+1} = L_i - x_0(L_i)\Delta L/[x_0(L_i+\Delta L)-x_0(L_i)], \quad \Delta L = 10^{-3} \tag{4.9}$$

where $i = 0, 1, 2, \ldots$ is the number of the iteration. Iterations (4.9) lead to a value $|x_0| < 10^{-6}$ after two to three steps.

As the modulus of the shear contact stress $P_{xz} = \sqrt{P_x^2 + P_z^2}$ approaches the limiting value of $1/2$, the field of the characteristics degenerates into a line, which coincides with the punch boundary. If $\varphi = \pi/2$ and $\theta \to \pi/2$, we obtain the longitudinal displacement of the punch along the z axis when $P_x = 0$ and $P_z = 1/2$. From Eqs (2.5) and (1.4), we find $\sigma = -2/3$, $\sigma_z = -1/2$. This is the case of pure shear with minimum pressure on the punch

$$P_y = 1/2, \quad P_x = 0, \quad P_z = 1/2 \quad \text{when} \quad \varphi = \theta = \pi/2 \tag{4.10}$$

Hence, when the shear contact stresses change, the limit pressure on the punch varies from a maximum value of $1 + \pi/2$ for a smooth Prandtl's punch to a minimum value of $1/2$ in the case of the pure longitudinal displacement of an absolutely rough punch.

The field of the characteristics for $\theta = 1$, $\varphi = 0.1563$ is shown in Fig. 1. On departing from the singular point A, the α- and β-characteristics are almost orthogonal since $\theta \to 0$ in accordance with the boundary conditions on AB. As the α- and β-characteristics approach the punch boundary, they become noticeably non-orthogonal since θ increases, approaching the value of unity, specified on OA.

An example of the field of the characteristics, when the shear contact stresses corresponding to the value $\theta = 1.25$, $\varphi = 0.588$ are increased; is shown in Fig. 3. Calculations show that the normal pressure on the punch is practically constant with an exceedingly small rise from the value of 1.781 at point O to 1.806 near the corner point A.

The mean limit pressures on the punch P_y, the shear contact stresses and the coordinates x_c of point C of the plastic region are shown in Table 1 for several values of the parameter θ.

The numerical results presented above were obtained for a flat punch at constant values of the shear contact stresses. The method of integrating hyperbolic differential equations for the general plane problem of the theory of ideal plasticity which has been developed can be used in the case of a non-uniform distribution of the shear contact stresses, a curvilinear boundary of the punch and a finite thickness of the workpiece, as applied to technological problems of the theory of plasticity [3, 4].

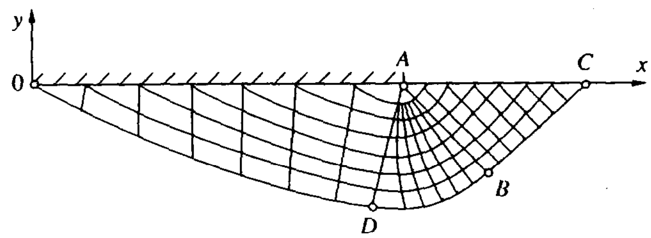

Fig. 3

Table 1

θ	0.5			1.0			1.25		
P_y	2.537	1.848	1.405	2.402	1.950	1.410	2.250	1.781	1.395
P_x	0.031	0.397	0.467	0.115	0.309	0.385	0.171	0.304	0.328
P_z	0.008	0.116	0.161	0.063	0.189	0.296	0.128	0.263	0.348
P_{xz}	0.032	0.414	0.495	0.131	0.362	0.486	0.214	0.402	0.478
x_c	1.977	1.421	1.105	1.891	1.549	1.178	1.802	1.481	1.229

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